Advanced Calculus Midterm Exam November 24, 2010

Note: There are 8 questions with total 126 points in this exam.

1. Let $\{x_k\}$ be a sequence defined recursively by $x_1 = \sqrt{2}$, and $x_{k+1} = \sqrt{2+x_k}$, for $k = 1, 2, \ldots$.
   (a) (10 points) Show by induction that (i) $x_k < 2$ and (ii) $x_k < x_{k+1}$ for all k.

   Solution: (i) When $k = 1$, $x_1 = \sqrt{2} < 2$.
   Assume that $x_k < 2$. Then we have $x_{k+1} = \sqrt{2+x_k} < \sqrt{2+2} = 2$.
   Thus, the argument of induction implies that $x_k < 2$ for all k.
   (ii) Using (i) twice, we have $x_{k+1} = \sqrt{2+x_k} > \sqrt{x_k + x_k} = \sqrt{2}\sqrt{x_k} > \sqrt{x_k}\sqrt{x_k} = x_k$ for all k.

   (b) (10 points) Show that $\lim\limits_{k\to\infty} x_k$ exists and evaluate it.

   Solution: The results of (a) implies that $\{x_k\}$ is an increasing sequence and it is bounded from above by 2. The Monotone convergence theorem implies that $\lim\limits_{k\to\infty} x_k = x$ exists, and since $x = \lim\limits_{k\to\infty} x_{k+1} = \lim\limits_{k\to\infty} \sqrt{2+x_k} = \sqrt{2+x}$, we have $0 = x^2 - x - 2 = (x-2)(x+1) \Longrightarrow x = 2$.

2. (16 points) Let $S \subset \mathbb{R}^n$ and $\mathbf{x} \in \mathbb{R}^n$. Show that $\mathbf{x} \in \overline{S}$ if and only if there is a sequence of points $\{\mathbf{x}_k\}$ in S that converges to $\mathbf{x}$. [**Hint:** You may use the fact that the closure of S is the union of S and all its boundary points, i.e. $\overline{S} = S \cup \partial S$.]

   Solution: $(\Rightarrow)$ If $\mathbf{x} \in \overline{S} = S \cup \partial S$, then either
   Case (i) : $\mathbf{x} \in S$, then the sequence $\{\mathbf{x}_k = \mathbf{x}\}$, for $k = 1, 2, \ldots$, is a sequence in S that converges to $\mathbf{x}$, or
   Case (ii) : $\mathbf{x} \in \partial S$, then, there is an $\mathbf{x}_k \in B(\frac{1}{k}, \mathbf{x}) \cap S$, for each $k = 1, 2, \ldots$ satisfying that $\lim\limits_{k\to\infty} \mathbf{x}_k = \mathbf{x}$.
   $(\Leftarrow)$ For each $\mathbf{x} \notin \overline{S}$, there exists an $r > 0$ such that $B(r, \mathbf{x}) \subset \overline{S}^c$. This implies there does not exist any sequence of points $\{\mathbf{x}_k\}$ in S that converges to $\mathbf{x}$.
   Remark: This equivalence says that $S \cup \partial S = S \cup S'$, where S' denote the set of accumulation points of S.

3. (a) (10 points) Let $B(r, \mathbf{0}) = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\| < r\}$ be the ball of radius r about the origin. Show that $B(r, \mathbf{0})$ is open in $\mathbb{R}^n$.

   Solution: For each $p \in B(r, \mathbf{0})$, we have $B(r - \|p\|, p) \subset B(r, \mathbf{0})$ since for each $y \in B(r - \|p\|, p)$ we have $\|y\| \le \|y - p\| + \|p\| < r - \|p\| + \|p\| = r$, i.e. $y \in B(r, \mathbf{0})$.

   Alternative proof: Since $\mathbf{0} \in B(r, \mathbf{0}) \subset B(r, \mathbf{0})$, $\mathbf{0}$ is an interior point of $B(r, \mathbf{0})$.
   For each $p \in B(r, \mathbf{0}) \setminus \{\mathbf{0}\}$, by setting $\rho = \min\{\|p\|, r - \|p\|\} > 0$, we have $B(\rho, p) \subset B(r, \mathbf{0})$ and conclude that p is an interior point of $B(r, \mathbf{0})$. Thus, $B(r, \mathbf{0})$ is open since each of its points is an interior point.

   (b) (16 points) Show that if S_1 and S_2 are open, so are $S_1 \cup S_2$ and $S_1 \cap S_2$.

   Solution: For each $p \in S_1 \cup S_2$, since p is an interior point of S_i, for $\text{i} = 1$ or 2, there exists a $B(r, p)$, for some $r > 0$, satisfying either $B(r, p) \subset S_\text{i} \subset S_1 \cup S_2$. Thus, p is an interior point of $S_1 \cup S_2$, and $S_1 \cup S_2$ is open.
   For each $p \in S_1 \cap S_2$, there exist $r_1, r_2 > 0$ such that $B(r_1, p) \subset S_1$, and $B(r_r, p) \subset S_2$. By taking $r = \min\{r_1, r_2\} > 0$, we have $B(r, p) \subset S_1 \cap S_2$, and conclude that $S_1 \cap S_2$ is open.

(c) (8 points) Show that for any $S \subset \mathbb{R}^n$, S^{int} is open.

Solution: For each $p \in S^{\text{int}}$, there is an $r > 0$ such that $B(r, p) \subset S$. By (a), every point in $B(r, p)$ is an interior point of $B(r, p) \subset S$, hence, it is also an interior point of S. Thus, $B(r, p) \subset S^{\text{int}}$, and this implies that S^{int} is open.
Alternative proof: The definition of an interior point says that $x \in S^{\text{int}}$ if there exists a ball $B(r,x) \subset S$. Therefore, if $x \notin S^{\text{int}}$, then either every ball centered at x intersects both S and S^c, or there exists a ball $B(r,x) \subset S^c$, i.e. $x \in \partial S \cup (S^c)^{\text{int}}$. Since $\partial S = \partial S^c$, $\partial S \cup (S^c)^{\text{int}} = \partial S^c \cup (S^c)^{\text{int}}$ is closed, S^{int} is open.

4. (10 points) Let $\mathbf{f} : S \to \mathbb{R}^m$ be a function satisfying

$$|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| \leq C|\mathbf{x} - \mathbf{y}|^\lambda \quad \text{for all } \mathbf{x}, \mathbf{y} \in S,$$

where $C > 0$ and $\lambda > 0$ are constants. Show that $\mathbf{f}$ is uniformly continuous on S.

Solution: For each $\varepsilon > 0$, since $|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| \leq C|\mathbf{x} - \mathbf{y}|^\lambda$ for all $\mathbf{x}, \mathbf{y} \in S$, we choose $\delta = (\frac{\varepsilon}{C})^{1/\lambda}$ such that if , $\mathbf{x}, \mathbf{y} \in S$ satisfying that $|\mathbf{x} - \mathbf{y}| < \delta$, then $|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| \leq C|\mathbf{x} - \mathbf{y}|^\lambda < C\delta = \varepsilon$. Hence, $\mathbf{f}$ is uniformly continuous on S.

5. (16 points) Show that $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^k$ is continuous (in the sense of ε-δ definition) if and only if for each open subset U in $\mathbb{R}^k$ the set $f^{-1}(U) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{f}(\mathbf{x}) \in U\}$ is open.

Solution: ($\Rightarrow$) Let U be an open subset in $\mathbb{R}^k$ and p be a point in $f^{-1}(U)$. Given $\varepsilon > 0$, since $B(\varepsilon, \mathbf{f}(p)) \cap U$ is open, there exists a ball $B(\eta, \mathbf{f}(p)) \subset B(\varepsilon, \mathbf{f}(p)) \cap U$, for some $\eta > 0$. The continuity of $\mathbf{f}$ at p implies that there is a ball $B(\delta, p)$, for some $\delta > 0$, such that $\mathbf{f}(B(\delta, p)) \subset B(\eta, \mathbf{f}(p)) \subset B(\varepsilon, \mathbf{f}(p)) \cap U \subset U$. This implies that $B(\delta, p) \subset \mathbf{f}^{-1}(U)$, and $\mathbf{f}^{-1}(U)$ is open .
($\Leftarrow$) For each $p \in \mathbb{R}^n$, and each $\varepsilon > 0$, since the set $B(\varepsilon, \mathbf{f}(p))$ is open in $\mathbb{R}^k$, $\mathbf{f}^{-1}(B(\varepsilon, \mathbf{f}(p)))$ is an open set containing p. There exists a $\delta > 0$ such that $B(\delta, p) \subset \mathbf{f}^{-1}(B(\varepsilon, \mathbf{f}(p)))$, i.e. $\mathbf{f}(B(\delta, p)) \subset B(\varepsilon, \mathbf{f}(p))$, and $\mathbf{f}$ is continuous at p.

6. (10 points) Suppose that $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ is continuous on $U \subset \mathbb{R}^n$ and $\mathbf{g} : \mathbb{R}^m \to \mathbb{R}^k$ is continuous on $\mathbf{f}(U) \subset \mathbb{R}^m$. Show that the composite function $\mathbf{g}(\mathbf{f}) : \mathbb{R}^n \to \mathbb{R}^k$ is continuous on U.

Solution: For each $p \in U$, and any $\varepsilon > 0$, since $\mathbf{g}$ is continuous at $\mathbf{f}(p)$, there exists $\delta_1 > 0$, such that if $y \in \mathbf{f}(U)$ satisfying that $|y - \mathbf{f}(p)| < \delta_1$, then $|\mathbf{g}(y) - \mathbf{g}(\mathbf{f}(p))| < \varepsilon$. Also, since $\mathbf{f}$ is continuous at p, there exists a $\delta > 0$, such that if $x \in U$ satisfying that $|x - p| < \delta$, then $|\mathbf{f}(x) - \mathbf{f}(p)| < \delta_1$ which implies that $|\mathbf{g}(\mathbf{f}(x)) - \mathbf{g}(\mathbf{f}(p))| < \varepsilon$, i.e. the composite function $\mathbf{g}(\mathbf{f})$ is continuous at each $p \in U$.

7. (10 points) Let S be a compact subset of $\mathbb{R}^n$ and let $\mathbf{f}: S \to \mathbb{R}^m$ be continuous at every point of S. Show that the image set

$$\mathbf{f}(S) = \{\mathbf{f}(\mathbf{x}) : \mathbf{x} \in S\}$$

is also compact.

Solution: Suppose $\{y_k\}$ is a sequence in $\mathbf{f}(S)$. This implies that, for each k, there is an $x_k \in S$ such that $y_k = \mathbf{f}(x_k)$. Since S is compact, by the Bolzano-Weierstrass theorem, $\{x_k\}$ has a convergent subsequence $\{x_{k_j}\}$ that converges to a point $a \in S$. Since $\mathbf{f}$ is continuous at a, $\lim_{j\to\infty} y_{k_j} = \lim_{j\to\infty} \mathbf{f}(x_{k_j}) = \mathbf{f}(a) \in f(S)$. Thus, every sequence in $\mathbf{f}(S)$ has a subsequence whose limit lies in $\mathbf{f}(S)$. This implies that $\mathbf{f}(S)$ is compact.

8. (10 points) Let S be a connected subset in $\mathbb{R}^n$. Show that the closure of S is also connected.

Solution: Suppose that $\overline{S}$ is disconnected and (U,V) is a disconnection of $\overline{S}$. Suppose that $U \cap S \neq \emptyset$ and $V \cap S \neq \emptyset$, then it is easy to see that $(U \cap S, V \cap S)$ is a disconnection of S. This contradicts to that S is connected. Therefore, either $U \cap S = \emptyset$, or $V \cap S = \emptyset$. Assume that $V \cap S = \emptyset$, since $S \cup \partial S = \overline{S} = U \cup V$, this implies that $V \subset \partial S$, and $U = S$. But, this implies that $\overline{U} \cap V = \overline{S} \cap V \neq \emptyset$ which contradicts to that (U,V) is a disconnection of $\overline{S}$. It is easy to see that $U \cap S = \emptyset$ will also lead to a contradiction. Therefore, $\overline{S}$ is connected.
Note: (1) In general, the converse is not true. e.g. Let $S = [0,1) \cup (1,2)$. Then $\overline{S} = [0,2]$ is connected while S is not.
(2) A subset $A \subset S$ is said to be **open relative to the set S** if there exists an open set $U \subset \mathbb{R}^n$ such that $A = U \cap S$.
The definition of connectedness of $S \iff$ is equivalent to that S cannot be a disjoint union of two nonempty open subsets relative to S, i.e. S cannot be expressed as $S = A \cup B$, where $\emptyset \neq A = U \cap S$, and $\emptyset \neq B = V \cap S$, $A \cap B = \emptyset$, and U,V are open subsets of $\mathbb{R}^n$.
proof of $(\Rightarrow)$ Suppose that $S = A \cup B$, where $\emptyset \neq A = U \cap S$, and $\emptyset \neq B = V \cap S$, $A \cap B = \emptyset$, and U,V are open subsets of $\mathbb{R}^n$.
$\Rightarrow A = S \setminus B = S \setminus (V \cap S) = S \setminus V = S \cap V^c$,
and $B = S \setminus A = S \setminus (U \cap S) = S \setminus U = S \cap U^c$.
$\Rightarrow \overline{A} \cap B = \left(S \cap V^c\right) \cap \left(S \cap V\right) = \emptyset$,
and $\overline{A} \cap B = \left(S \cap V^c\right) \cap \left(S \cap V\right) = \emptyset$.
Hence, S is disconnected.
proof of $(\Leftarrow)$ Suppose that S is disconnected, and $S = S_1 \cup S_2$,
where $\emptyset \neq S_i$, $i = 1,2$, and $\overline{S_1} \cap S_2 = \emptyset$, $S_1 \cap \overline{S_2} = \emptyset$.
$\Rightarrow S_1 = \overline{S_2}^c \cap S$, and $S_2 = \overline{S_1}^c \cap S$ are disjoint nonempty open subsets relative to S, and $S = S_1 \cup S_2$.